Exam: Introduction to Condensed Matter Theory

Wednesday, April 14, 2010

The total number of points is 50. Do not forget to write your name and the student number on the first sheet. Good luck.

1. **Elastic energy of a tetragonal crystal:** Find the most general expression for the elastic energy of a tetragonal crystal in terms of the components of the strain tensor u_{ik}. [10 points]

   Hint: A tetragonal crystal has symmetries of a rectangular cuboid with $a = b \neq c$ (see Fig. 1).

2. **Temperature dependence of magnetization:** Consider a magnet where spins form a cubic lattice with the site coordinates $\mathbf{X}_n = a\left(n_x\hat{x} + n_y\hat{y} + n_z\hat{z}\right)$. The Hamiltonian describing interactions between neighboring spins along the x, y and z directions has the form

$$H = -J\sum_n \left[\mathbf{S}_n\cdot\mathbf{S}_{n+x} + \mathbf{S}_n\cdot\mathbf{S}_{n+y} + \mathbf{S}_n\cdot\mathbf{S}_{n+z}\right], \tag{1}$$

   where $J > 0$ is the exchange constant ($n+x$ denotes the site with the coordinate $\mathbf{X}_n + a\hat{x}$). At nonzero temperatures the average value of the z-projection of spin in the ferromagnetic state, $\langle S_n^z\rangle$, is smaller than its maximal value equal $S \gg 1$, reached at $T = 0$. At low temperatures, $k_BT \ll 12JS$,

$$S - \langle S_n^z\rangle \propto T^{\alpha}. \tag{2}$$

   Find the exponent α. [10 points]

   Hint: Use the Dyson-Maleev transformation:

$$\begin{aligned} S^- = S^x - iS^y &= \sqrt{2S}a^\dagger, \\ S^+ = S^x + iS^y &= \sqrt{2S}\left(1 - \frac{a^\dagger a}{2S}\right)a, \\ S^z = S - a^\dagger a. & \end{aligned} \tag{3}$$

3. **Effective electron mass:** Consider electrons on a two-dimensional triangular lattice with the lattice constant a, described by the tight-binding Hamiltonian,

$$H = -t\sum_{\langle n,m\rangle}\sum_{\sigma}\left(c_{m\sigma}^\dagger c_{n\sigma} + c_{n\sigma}^\dagger c_{m\sigma}\right), \tag{4}$$

where $\langle n,m\rangle$ denotes a pair of nearest-neighbor sites of the triangular lattice and t is the hopping amplitude. Show that

$$\varepsilon(\mathbf{k}) \approx \varepsilon(0) + \frac{\hbar^2 k^2}{2m_*}, \quad \text{for} \quad ka \ll 1. \tag{5}$$

where $\varepsilon(\mathbf{k})$ is the energy of electron with the wave vector $\mathbf{k}$, $k^2 = k_x^2 + k_y^2$ and m_* is the effective electron mass. For $t = 0.5$eV and $a = 3$Å, find $\frac{m_*}{m_e}$, where m_e is the electron mass in vacuum. [10 points]

4. **Dielectric susceptibility:** Consider a hypothetic material whose optical absorption spectrum contains a single sharp peak at the frequency Ω:

$$\chi''(\omega) = A\delta(\omega - \Omega), \quad \text{for} \quad \omega > 0, \tag{6}$$

where $\chi''(\omega)$ is the imaginary part of the dielectric susceptibility. Find the real part of the dielectric susceptibility, $\chi'(\omega)$, of this material and draw a plot of χ' versus frequency. [10 points]

Hint: Use Kramers-Kronig relations.

5. **Magnetic levitation:** Figure 2 shows a magnet levitating above a superconductor. Explain the physical origin of this effect. Which force counteracts gravity and what determines the distance from the magnet to the surface of the superconductor? [10 points]

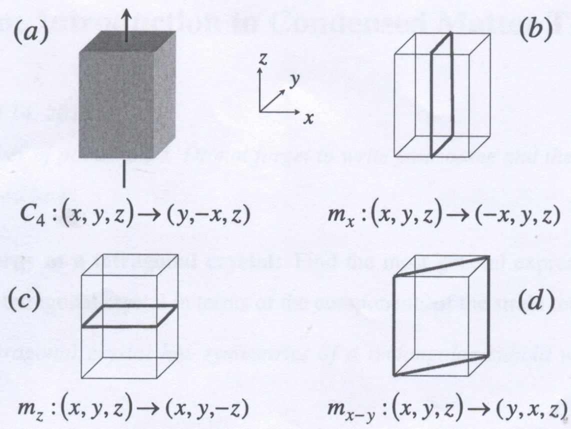

FIG. 1: Some tetragonal symmetries: (a) 90° rotation around the z axis, (b) reflection in the yz plane, (c) xy mirror plane (d) mirror interchanging the x and y axes. All symmetry operations of a tetragonal crystal can be obtained by combining C_4, m_x, m_z and their inverse.

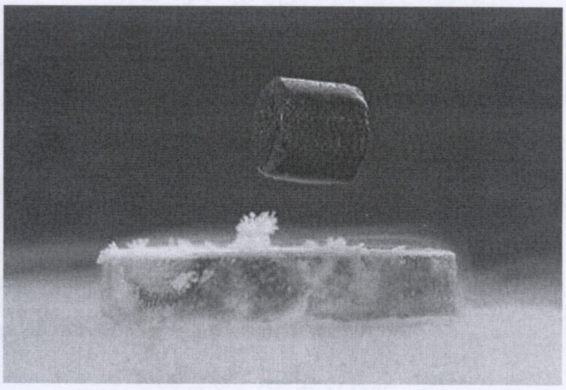

FIG. 2: A magnet levitating above a superconductor.